Solutions to Quiz 1, ECED 3300, 2019

Problem 1

By definition,

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial_x & \partial_y & \partial_z \\ x + c_1 z & z + c_2 y & y + c_3 x \end{vmatrix} = \mathbf{a}_x(1-1) - \mathbf{a}_y(c_3 - c_1) + 0\mathbf{a}_z$$

The vector equation $\nabla \times \mathbf{A} = 0$ implies that $c_3 = c_1$. Thus, whenever $c_1 = c_3$ and for arbitrary c_2, the field is irrotational, i.e, it has a zero curl everywhere.

Problem 2

Let us first work out the flux of $\mathbf{F} = \mathbf{a}_r$ over the surface of a unit sphere. In this case, $\mathbf{a}_n = \mathbf{a}_r$ and $d\mathbf{S} = \mathbf{a}_r \sin\theta d\theta d\phi$ because on the sphere $r = 1$. It follows that

$$\oint d\mathbf{S} \cdot \mathbf{F} = \int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} d\phi(\mathbf{a}_r \cdot \mathbf{a}_r) = 2\pi(-\cos\theta)|_0^\pi = \underline{4\pi}. \tag{1}$$

On the other hand,

$$\nabla \cdot \mathbf{F} = \tfrac{1}{r^2}\partial_r(r^2) = \tfrac{2}{r}.$$

It follows that

$$\int dv \nabla \cdot \mathbf{F} = \underbrace{\int_0^1 dr \, r^2 \times \left(\tfrac{2}{r}\right)}_{=1} \underbrace{\int_0^\pi d\theta \sin\theta}_{=2} \underbrace{\int_0^{2\pi} d\phi}_{=2\pi} = \underline{4\pi}. \tag{2}$$

It can be inferred from Eqs. (1) and (2) that

$$\oint d\mathbf{S} \cdot \mathbf{F} = \int dv \nabla \cdot \mathbf{F},$$

thereby verifying Gauss's theorem.

Problem 3

a) Using the vector identity,

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \equiv (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}),$$

we obtain,

$$f = (\mathbf{a}_z \times \mathbf{r}) \cdot (\mathbf{a}_z \times \mathbf{r}) = \underbrace{(\mathbf{a}_z \cdot \mathbf{a}_z)}_{=1}(\mathbf{r} \cdot \mathbf{r}) - (\mathbf{a}_z \cdot \mathbf{r})^2 = r^2 - z^2 = x^2 + y^2.$$

It follows at once that

$$\nabla f = \mathbf{a}_x \partial_x f + \mathbf{a}_y \partial_y f = 2x\mathbf{a}_x + 2y\mathbf{a}_y = 2(x\mathbf{a}_x + y\mathbf{a}_y).$$

b) In the cylindrical coordinates,

$$\nabla f = 2(x\mathbf{a}_x + y\mathbf{a}_y) = 2\rho\mathbf{a}_\rho.$$

To see that, it is sufficient to read off from the Formula Sheet, $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $\mathbf{a}_x \cos\phi + \mathbf{a}_y \sin\phi = \mathbf{a}_\rho$.